Intelligent Systems: Reasoning and Recognition

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Bayes Rule with Probability Distributions and Densities

Notation

E	An observation or event, typically with feature value X or $\vec{X}$
x	A variable
X	A random variable (unpredictable value). an observation.
N	The number of possible values for X
$\vec{x}$	A vector of D variables.
$\vec{X}$	A vector of D random variables.
D	The number of dimensions for the vector $\vec{x}$ or $\vec{X}$
N_d	The number of values (or symbols) for x_d (The d^{th} component of $\vec{X}$)
$h(\vec{x})$	A D dimensional frequency table for values of $\vec{X}$
Q	The number of cells in $h(\vec{x})$. $Q=N_1 \cdot N_2 \cdot \ldots \cdot N_D$
C_k	The class k
k	Class index
K	Total number of classes
ω_k	The statement (assertion) that $X \in C_k$
$P(X \in C_k)$	Probability that the observation X is a member of the class k.
M_k	Number of examples for the class k.
M	Total number of examples. $M = \sum_{k=1}^{K} M_k$
$\{\vec{x}_m\}$	A set of training samples
$\{y_m\}$	A set of indicator vectors for the training samples in $\{\vec{X}_m\}$
$p(X)$	Probability density function for a continuous value X
E_{RMS}	Mean square error for using $\frac{1}{M}h(x)$ as an estimation for $p(X)$
$E\{X_m\}$	The expected value of x: $E\{x_m\} = \frac{1}{M}\sum_m x_m$
O(-)	Order of operator, (also called growth rate or fit approximation.) Example: $f(n) = 4n^3+3n^2+10$ then $O(f(n)) = n^3$

Probability

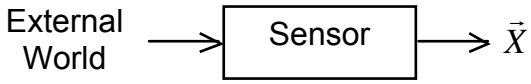
There are two possible definitions of probability that we can use for reasoning and recognition: Frequentialist and Axiomatic.

Probability as Frequency of Occurrence

A frequency-based definition of probability is sufficient for many practical problems.

In the following we will use the symbol E to refer to an observation, to avoid confusion with X, the value of features that describe an observation. It is possible to have two observations E_1, and E_2 that are described by identical feature values X. For example, two individuals may both have blue eyes.

Assume that we have some form of sensor that generates observations, E, at that each observation belongs to one and only one of K classes, $\{C_k\}$. The class for each observation is "random". This means that the exact class cannot be predicted in advance.



Suppose we have a set of M observations $\{E_m\}$, for which M_k of these events belong to the class C_k. The probability that any observed events from the set $\{E_m\}$ belongs to the class C_k is the relative frequency of occurrence of the class C_k in the set $\{E_m\}$.

The probability that E_m belongs to C_k is $P(E_m \in C_k) = \frac{M_k}{M}$

If we make new observations under the same condition, it is reasonable to expect the fraction to be the similar or the same. However, because the observations are random, there may be differences. This is a form of sampling error. These differences will grow smaller as the size of the set of observations, M, grows larger.

A frequency-based definition is easy to understand and can be used to build practical systems. It can also be used to illustrate basic principles. However it is possible to generalize the notion of probability with an axiomatic definition. This will make it possible to apply probability theory in cases where the frequency-based interpretation is not obvious.

Axiomatic Definition of probability

An axiomatic definition of probability makes it possible to apply analytical techniques to the design of reasoning and recognition systems. Only three postulates (or axioms) are necessary:

In the following, let E be an observation (or event), let $S=\{E\}$ be a set of observations, and let $\{C_k\}$ be a set of K disjoint (non-overlapping) subsets of observations from S. That is, $\forall C_i, C_j \subset S : C_i \cap C_j = \varnothing$

Any function *P(-)* that obeys the following 3 axioms can be used as a probability:

For any observation (event) E:
axiom 1: $P(E \in C_k) \geq 0$ (Probability is a positive number)
axiom 2: $P(E \in S) = 1$ (Every event E belongs to the set of all events, S)
axiom 3 : $\forall C_i, C_j \subset S$ such that $C_i \cap C_j = \varnothing$: $P(E \in C_i \cup C_j) = P(E \in C_i) + P(E \in C_j)$
(for disjoint subsets, probabilities add).

An axiomatic definition of probability can be very useful if we have some way to estimate the relative "likelihood" of different propositions.

Let us define ω_k as the proposition that an observation E, belongs to class C_k:
$\omega_k \equiv E \in C_k$

The likelihood of the proposition, $L(\omega_k)$, is a numerical function that estimates of its relative "plausibility" or believability of the proposition.

Assuming that $L(\omega_k)$ obeys axioms 1 and 3, we can convert likelihoods into probabilities by normalizing so that the sum of all likelihoods is 1. To do this we simply divide by the sum of all likelihoods:

$$P(\omega_k) = \frac{L(\omega_k)}{\sum_{k=1}^{K} L(\omega_k)}$$

Normalisation assures that axiom 2 is respected.
We will examine three different representations for probability: Distribution Tables, Histograms and Density functions. Each of these can be used with Bayes rule to infer the probability that an event belongs to a class.

Bayes' Rule

Bayes' rule provides a unifying framework for pattern recognition and for reasoning about hypotheses under uncertainty. "Bayes" refers to the 18th century mathematician and theologian Thomas Bayes (1702–1761), who provided the first mathematical treatment of a non-trivial problem of Bayesian inference. Bayesian inference was made popular by Simon Laplace in the early 19th century.

The rules of Bayesian inference can be interpreted as an extension of logic. Many machine-learning methods are based on Bayesian principles.
Bayes Rule gives us a tool to reason with conditional probabilities.
Conditional probability is the probability of an event given that another event has occurred. Conditional probability measures "correlation" or "association".

Consider two non-disjoint classes (sets) of events A and B.
Let P(A) be the probability that an event $E \in A$
Let P(B) be the probability that an event $E \in B$ and
Let P(A, B) be the probability that the event is in both A and B.

We can note that $P(A, B) = P((E \in A) \wedge (E \in B)) = P(E \in A \cap B)$
We write this as : $P(A, B) = P(A \wedge B) = P(A \cap B)$

Conditional probability can be defined as $P(A \mid B) = \frac{P(A,B)}{P(B)}$

Equivalently, conditional probability can be defined as

$$P(A \mid B)P(B) = P(A,B)$$

The union of sets is commutative, so $P(A,B) = P(B,A) = P(B \mid A)P(A)$

This gives a common definition of Bayes Rule:

$$\boxed{P(A \mid B)P(B) = P(A,B) = P(B \mid A)P(A)}$$

This can be generalized to more than 2 classes, giving, for example,

$$P(A,B,C) = P(A \mid B,C)P(B,C) = P(A \mid B,C)P(B \mid C)P(C)$$

To use these tools we need techniques to represent and compute probability.

Probability Distribution Tables

A **Probability Distribution Table** that gives the relative frequency of occurrence for all possible values of a property (or feature) for a set of observations (or events).

Suppose that we have a set of M observations that can be divided into N subsets, such that the subsets are (1) Mutually Exclusive and (2) Complete. These subsets represent "values", X, for the observation. (sometimes called Feature Values).

Feature values are mutually exclusive and the set of values is complete.
A single observation has a unique value, X. The value of an observation must be from the set of possible values.

Features values can be binary, or numeric (natural, integer or real), or symbolic (category labels). We will start by illustrating this with symbolic features.

A set of M people can be divided into subsets defined by eye color:
{Blue, Green, Brown, Hazel, Gray}.
This set is (1) Mutually Exclusive and (2) Complete.

A Probability Distribution Table that gives the relative frequency of occurrence for each value of a feature for a set of observations.

Let X represent the Eye Color $X \in$ {Brown, Blue, Green, Hazel, Gray}, $N=5$.

Let $h(x)$ be a table of 5 counters for the values of X. $h(x)$ is initially 0.
The table $h(x)$ can be easily implemented using the symbols as a key that indexes into h(x) to give a value. The keys are the labels: {Brown, Blue, Green, Hazel, Gray}
The values of $h(x)$ are the number of observations with each label.

Capital X is a random variable (variable with unknown value) for the set of labels, lower case x is a specific value of X.

Suppose that we have a set, S, of M people, such that X_m is the eye color of person m.

For each person we will increment the table $h(X_m) \leftarrow h(X_m) + 1$

Formally: $\forall m = 1, M : \; h(X_m) \leftarrow h(X_m) + 1$

Note that because each person can have one and only one feature value:

$$M = \sum_x h(x)$$

The probability distribution table gives the probability that a person *m* has eye-color *x*. This can be computed from:

$P(X_m = x) = \frac{1}{M} h(x)$ This is commonly written: $P(X) = \frac{1}{M} h(X)$

To be a valid probability, the values must sum to 1: $1 = \sum_x P(x)$

The most probable feature value (most likely eye color) is the feature value with the highest probability

$$\hat{X} \leftarrow \arg\max_x \{P(x)\}$$

This is a property of the set of m persons and not of the individuals of the set.

Joint Probability Distributions Tables (PDTs)

Distribution tables can be generalized to multiple features.
For example, the persons in the set S have a gender, G, as well as Eye color, C.
Suppose we have M=100 students.

Let G_m represent the gender of each person $G_m \in \{\text{Male, Female}\}$, $N_G=2$
and let C_m represent the Eye-color of student.

$C_m \in \{\text{Brown, Blue, Green, Hazel, Grey}\}$, $N_c=5$ (5 possible colors)

A two dimensional table, h(c,g) counts the number of times that a student, S_m, of Gender G_m has eye color C_m

$$\forall m = 1, M: \; h(C_m, G_m) \leftarrow h(C_m, G_m) + 1$$

The number of cells in the h(c,g) is $N_C \cdot N_G = 2 \times 5 = 10$.
For example, suppose that we have 100 students with the following histogram

h(c,g)	Brown	Blue	Green	Hazel	Gray
Male	64	8	4	3	1
Female	14	5	1	0	0

The joint probability is the frequency of occurrence for each pair C and G.

$$P(C_m = c \wedge G_m = g) = \frac{1}{M} h(c,g)$$

Thus the **joint probability distribution table** for gender and eye color is obtained by dividing by the histogram $h(c,g)$ by the number of students, M.

P(c,g)	Brown	Blue	Green	Hazel	Gray
Male	0.64	0.08	0.04	0.03	0.01
Female	0.14	0.05	0.01	0	0

The complete table must sum to 1. $\sum_{c \in C} \sum_{g \in G} P(c,g) = 1$

We can eliminate a class from the table by summing a row or column:

$$P(C) = \sum_{g \in G} P(C,g) \quad \text{and} \quad P(G) = \sum_{x \in C} P(x,G)$$

For example, independent of gender, the probability of eye color is

P(c)	Brown	Blue	Green	Hazel	Gray
-	0.78	0.13	0.05	0.03	0.01

The probability for each gender is

Gender	P(G)
Male	0.8
Female	0.2

The joint probability is the product $P(C,G) = P(C) \cdot P(G)$

All this can be generalized to any number of features. For three features A, B, C

$$P(A,B,C) = \frac{1}{M} h(A,B,C) \quad \text{and} \quad P(A,B) = \sum_{x \in C} P(A,B,x)$$

Normally eye color is independent of gender. However, in this example blue-eyed females are more likely than blue eyed males. This is the result of "sample error".

Let Q be the number of cells in h(). $Q = N_c \cdot N_G$

RMS Sample error, E_{RMS} is proportional to the ratio Q/M: $E_{RMS} \approx \frac{Q}{M}$

As a rule of thumb, it is recommended that $M \geq 10Q$.

Conditional Probability Tables (CPTs)

Bayes Rule can be expressed with conditional probability tables. For a probability distribution *P(A,B)* the Conditional probability can be defined as

$$P(A \mid B) = \frac{P(A,B)}{\sum_x P(x,B)} = \frac{P(A,B)}{P(B)}$$

With multiple features; $$P(A,B \mid C) = \frac{P(A,B,C)}{\sum_{x \in C} P(A,B,x)} = \frac{P(A,B,c)}{P(A,B)}$$

For example, consider the question: Will it rain today?

Rain can be predicted from atmospheric pressure. Rain is very unlikely when the pressure is high, possible when the pressure is normal, and likely when the pressure is low. We can represent intervals of pressure using the symbols L, N, and H based on measured pressure in hecto-Pascals:

L: P ≤ 1005 hP
N: 1005 hP ≤ P < 1020 hp
H: P ≥ 1020 hp

Rain (R) can be predicted from categories (or intervals) of atmospheric pressure (P) using a **Conditional Probability Table (CPT)**:

P(R|P)	R	¬R
L	0.8	0.2
N	0.3	0.7
H	0.01	0.99

The CPT gives the probability of (R) for each possible value of pressure (P). Note that the rows must sum to 1, but not the columns. When R is a binary value, many authors will ignore the last column.

This table can be built up from historical records for a location, noting the frequency of rain for each region of pressure.

To predict rain we need to predict pressure. Weather forecasters use weather models to predict the movement of high and low pressure cells. From this they obtain a prediction of atmospheric pressure for a give location over a future interval of time.

This may be expressed as a probability distribution table:

P(P)	L	N	H
-	0.1	0.9	0

Then probability of rain may be computed from

$$P(R) = \sum_{p \in \{L,N,H\}} P(R \mid p) \cdot P(p)$$

Giving: P(R) = 0.8 (0.1)+ 0.3(0.9) +0.01 (0) = 0.35
P(¬R)=0.2(0.1)+0.7(0.9)+0.99(0) =0.65

Thus:

P(R)	Rain	¬Rain
	0.6	0.4

Note that most authors would only compute P(R). Here we have computed P(¬R) to illustrate that the probabilities sum to 1.

In our example, we used rough categories (L, N, H) for atmospheric pressure. We can be more accurate by expanding the number of categories. For example, we can create one category for each integer value of pressure in HectoPascal, within the range of historically observed pressures 870 hP to 1083 hP.

This would allow us to use the numerical value of pressure as an integer index, rather than a symbolic key. In this case the Probability Distribution Table and Conditional Probability Tables can be computed using histograms of historically recorded values.

Histograms for Numerical Properties

The notion of probability and frequency of occurrence are easily generalized to describe the likelihood of numerical properties (features), X, observed by sensors.

For example, consider the height, measured in cm, of people present in this lecture today. Let us refer to the height of each student m, as X_m.

We can generate a histogram, $h(x)$, for the M students present.
For convenience we will treat height as an integer from the range 151 to 250.
We will allocate a table $h(x)$, of 100 cells. The size of the table is Q=100

The number of cells is called the capacity of the histogram, Q

We then count the number of times each height occurs in the class.

$$\forall m=1, M : h(x_m) := h(x_m) + 1;$$

After counting the heights we can make statements about the population of students. For example, the relative likelihood of height that a random student has a height of X=180cm is

$$L(X=180) = h(180)$$

This is converted to a probability by normalizing so that the values of all likelihoods sum to 1 (axiom 2).

$$P(X = x) = \frac{1}{M} h(x) \quad \text{where} \quad M = \sum_{x=151}^{250} h(x)$$

However, for this to be valid we need to have more data than the number of histogram cells: As we noted above, RMS Sample error, E_{RMS} is proportional to the ratio Q/M: $E_{RMS} \approx \frac{Q}{M}$

This gives a general rule of $M > 10\ Q$ predicting an $E_{RMS} \leq 0.10$

The actual RMS error of using a distribution table for a probability depends on the nature of the probability that is being estimated.
$E_{RMS} = 0.10$ for $M = 10\ Q$ is a worst case.

Bayes Rule with a Ratio of Histograms

Consider an example of K classes of objects where objects are described by a feature, X, with N possible integer values from [1, N]. Assume that we have a "training set" of M samples $\{x_m\}$ along with indicator variables $\{y_m\}$ where the indicator variable is the class, k, for each training sample.

For each class k, we allocate a histogram, $h_k()$, with N cells and count the values in the training set.

$$\forall_{m=1}^{M}: \; h(X_m) \leftarrow h(X_m) + 1$$
$$\text{IF } y_m = k \text{ THEN}$$
$$h_k(X_m) \leftarrow h_k(X_m) + 1;$$
$$M_k \leftarrow M_k + 1$$

Then

$$P(X = x \mid X \in C_k) = P(X \mid \omega_k) = \frac{1}{M_k} h_k(x)$$

The histogram for all possible values is the sum of histograms:

$$P(X = x) = \frac{1}{M} h(x) = \frac{1}{M} \sum_k h_k(x)$$

and $P(\omega_k)$ can be estimated from the relative number of events in each class.

$$P(X \in C_k) = P(\omega_k) = \frac{M_k}{M}$$

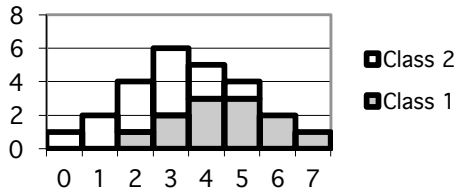
Bayes rule tells us: $P(A \mid B) = \dfrac{P(B \mid A)P(A)}{P(B)}$

giving: $$P(\omega_k \mid X) = \frac{P(X \mid \omega_k)P(\omega_k)}{P(X)} = \frac{\frac{1}{M_k} h_k(X) \frac{M_k}{M}}{\frac{1}{M} h(X)} = \frac{h_k(X)}{h(X)}$$

This can also be written as: $$P(\omega_k \mid X) = \frac{h_k(X)}{\sum_{k=1}^{K} h_k(X)}$$ because $$h(X) = \sum_{k=1}^{K} h_k(X)$$

The ratio of histograms can be represented by a lookup table. $P(\omega_k \mid X) = T(X)$

To illustrate, consider an example with 2 classes (K=2) and where X can take on 8 values (N=8, D=1).



Note that having M >> Q is NECESSARY but NOT Sufficient.
Having M < Q is a guarantee of INSUFFICIENT TRAINING DATA.

Number of samples required

Problem: Given a feature *x*, with N possible values, how many observations, M, do we need for a histogram, *h(x)*, to provide a reliable estimate of probability?

The worst case Root Mean Square error is proportional to $O(\frac{Q}{M})$.

This can be estimated by comparing the observed histograms to an ideal parametric model of the probability density or by comparing histograms of subsets samples to histograms from a very large sample. Let *p(x)* be a probability density function. The RMS (root-mean-square) sampling error between a histogram and the density function is

$$E_{RMS} = \sqrt{E\left\{\left(\frac{1}{M}h(x) - p(x)\right)^2\right\}} \approx O(\frac{Q}{M})$$

The worst case occurs for a uniform probability density function.

For most computational applications, M ≥ 8 Q (8 samples per "cell") is reasonable (less than 12% RMS error) and allows us to reason in powers of 2 2^3=8 ≈ 10

So what can you do if you do not have M >> Q ?
Adapt the size of the cell to the data!

Mean and Standard Deviation

An important difference is that with numerical values and symbolic categories is that numerical values obey an order relation: $1 < 2 < 3$, and a distance metric. $\forall x : dist(x, x+1) = 1$

With symbolic categories such as Blue, Green and Brown there is no intrinsic order and no metric for distance. (How far is Green from Brown?)

We can use this to make statements about the population of students in the class:

Consider a table h(x) of the height of the students in this class.

1) The average height of a member of the class is:

$$\mu_x = E\{x_m\} = \frac{1}{M}\sum_{m=1}^{M} x_m = \frac{1}{M}\sum_{x=x_{\min}}^{x_{\max}} h(x)\cdot x$$

Note that the average is the first moment, or center of gravity, of the histogram.

2) The variance is the square of the average difference from the mean:

$$\sigma_x^2 = E\{(x_m - \mu_x)^2\} = \frac{1}{M}\sum_{m=1}^{M}(x_m - \mu_x)^2 = \frac{1}{M}\sum_{x=1}^{250} h(x)\cdot(x - \mu_x)^2$$

The average difference from the mean, σ_x, is called the "standard deviation", and is often abbreviated "std." In French we call this the "écart type".

Average and variance are properties of the sample population.

Histograms with integer and real valued features

If X is an integer value then we need only bound the range to use a histogram

If ($x < x_{min}$) then $x := x_{min}$;
If ($x > x_{max}$) then $x := x_{max}$;

Then allocate a histogram of $N=x_{max}$ cells.

We may, for convenience, shift the range of values to start at 1, so as to convert integer x to a natural number:

$$n := x - x_{min} + 1$$

This will give a set of $N = x_{max} - x_{min} + 1$ possible values for X.

If X is real-valued and unbounded, we can limit it to a finite interval and then quantize with a function such as "trunc()" or "floor()". The function trunc() returns the largest integer that is less than or equal to a floating point (real) argument.

To quantize a real X to N discrete natural numbers : [1, N]

If ($X < x_{min}$) then $X := x_{min}$;
If ($X > x_{max}$) then $X := x_{max}$;

$$n = trunc\left(N \cdot \frac{X - x_{\min}}{x_{\max} - x_{\min} + 1} \right) + 1$$

Histograms for Vectors of Properties

We can also generalize to multiple properties. For example, each person in this class has a height, weight and age. We can represent these as three integers x_1, x_2 and x_3. Thus each person is represented by the "feature" vector $\vec{X} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$.

We can build up a 3-D histogram, $h(x_1, x_2, x_3)$, for the M persons in this lecture as:

$$\forall m = 1, M : h(\vec{x}_m) = h(\vec{x}_m) + 1$$

or equivalently: $\forall m=1, M : h(x_1, x_2, x_3) := h(x_1, x_2, x_3) + 1;$

and the probability of a specific vector is $P(\vec{X} = \vec{x}) = \frac{1}{M} h(\vec{x})$

When each of the D features can have N values, the total number of cells in the histogram will be $Q = N^D$

Probability Density Functions

A probability density function (PDF) is:
A probability density function *p(X)*, is a function of a continuous variable *X* such that

1) *X* is a continuous real valued random variable with values between $[-\infty, \infty]$
2) $\int_{-\infty}^{\infty} p(X) = 1$

Note that *p(X)* is NOT a number but a continuous function.

A probability density function defines the relatively likelihood for a specific value of *X*. Because *X* is continuous, the value of *p(X)* for a specific *X* is infinitely small. To obtain a probability we must integrate over some range of *X*.
To obtain a probability we must integrate over some range V of X.
In the case of D=1, the probability that X is within the interval [A, B] is

$$P(X \in [A,B]) = \int_{A}^{B} p(x)dx$$

This integral gives a number that can be used as a probability.

Note that we use upper case $P(X \in [A,B])$ to represent a probability value,
and lower case *p(X)* to represent a probability density function.

Bayes Rule with probability density functions
Classification using Bayes Rule can use probability density functions

$$P(\omega_k \mid X) = \frac{p(X \mid \omega_k)}{p(X)} P(\omega_k) = \frac{p(X \mid \omega_k)P(\omega_k)}{\sum_{j=1}^{K} p(X \mid \omega_j)P(\omega_j)}$$

Note that the ratio $\frac{p(X \mid \omega_k)}{p(X)}$ IS a number, provided that $p(X) = \sum_{k=1}^{K} p(X \mid \omega_k)P(\omega_k)$

Probability density functions are easily generalized to vectors of random variables.
Let $\vec{X} \in R^D$, be a vector random variables.
A probability density function, $p(\vec{X})$, is a function of a vector of continuous variables
1) $\vec{X}$ is a vector of D real valued random variables with values between $[-\infty, \infty]$
2) $\int_{-\infty}^{\infty} p(\vec{x})d\vec{x} = 1$

We concentrate on the Gaussian density function.