

# **Image Analysis and Formation (Formation et Analyse d'Images)**

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Lesson 1

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## Homogeneous Coordinates

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# 1. Homogeneous Coordinates and Tensor Notation

Homogeneous coordinates allow us to express translation, rotation, scaling, and projection all as matrix operations. The principle is to add an extra dimension to each vector.

For example, points on a plane are expressed as:

$$\bar{P} = \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

Similarly, points in 3D space become

$$\bar{Q} = \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

The line equation,  $ax+by+c=0$  can be expressed as a simple product:

$$\bar{L}^T \bar{P} = (a \ b \ c) \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = 0 \quad \text{where} \quad \bar{L}^T = (a \ b \ c)$$

Similarly, for a plane equation:  $ax+by+cz+d=1$ :

$$\bar{S}^T \bar{P} = (a \ b \ c \ d) \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = 0 \quad \text{where} \quad \bar{S} = (a \ b \ c \ d)$$

Note that in Homogeneous coordinates, all scalar multiplications are equivalent.

$$a \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = b \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

Any vector can be expressed in "canonical" form by normalizing the last coefficient to 1.

$$\begin{pmatrix} ax \\ ay \\ a \end{pmatrix} = \begin{pmatrix} ax/a \\ ay/a \\ a/a \end{pmatrix} = \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

Our camera model will have the form of a 3 x 4 matrix

$$M_s^i = \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \end{pmatrix}$$

such that

$$\begin{pmatrix} q_1/q_3 \\ q_2/q_3 \\ 1 \end{pmatrix} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \bar{Q} = M_s^i \bar{P} = \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \end{pmatrix} \begin{pmatrix} x_s \\ y_s \\ z_s \\ 1 \end{pmatrix}$$

We can express this in "canonical form" by dividing out the last coefficient:

$$M_s^i = \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & 1 \end{pmatrix}$$

Notice that this gives 11 coefficients. This corresponds to our 11 parameters. The Rank of M is 11.

## Tensor Notation:

In tensor notation, the sign " $\vec{\phantom{a}}$ " is replaced by subscripts and superscripts.

A super-script signifies a column vector.

For example the point  $\bar{P}$  is  $P^i$

$$P^i = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}$$

The line  $\bar{L}^T$  is  $L_i = (l_1, l_2, l_3)$

A matrix is a line matrix of column matrices (or a column of line matrices)

$$M_i^j = \begin{pmatrix} m_1^1 & m_2^1 & m_3^1 \\ m_1^2 & m_2^2 & m_3^2 \\ m_1^3 & m_2^3 & m_3^3 \end{pmatrix}$$

When homogeneous coordinates are used to represent transforms, these indices can be used to indicate the reference frame.

For example: A transformation from the scene "s" to the image "i" is a 3 x 4 matrix M

$$M_s^i = \begin{pmatrix} m_1^1 & m_2^1 & m_3^1 & m_4^1 \\ m_1^2 & m_2^2 & m_3^2 & m_4^2 \\ m_1^3 & m_2^3 & m_3^3 & 1 \end{pmatrix}$$

The sub/super scripts indicate the source and destination reference frames.  $M_s^i$  is a transformation from "S" (Scene) to "i" (image).

Einstein summation convention:

The summation symbol is implicit when a superscript and subscript have the same letter.

$$L_i P^i = l_1 p^1 + l_2 p^2 + l_3 p^3$$

for a matrix and vector, this gives a new vector:

$$p^j = T_i^j p^i$$

This example transforms the point  $p$  in reference  $i$  to a point  $p$  in reference  $j$ .

## 2. Cross Products

### Two point make a line

A line is a set of points that satisfies a constraint. The constraint is the line equation.

$$\vec{L}^T \vec{P} = ax + by + c = 0$$

The constraint can be determined from any two points on the line  $P_1 = (x_1, y_1)$  and  $P_2 = (x_2, y_2)$ .

Classically, we substitute  $P_1$  and  $P_2$  in the line equation to obtain two equations

$$\begin{aligned} ax_1 + by_1 + c &= 0 \\ ax_2 + by_2 + c &= 0 \end{aligned}$$

We then solve the two equations for a, b and c. This should give :

$$\begin{aligned} a &= (y_1 - y_2) & b &= (x_2 - x_1) \\ c &= -(a x_1 - b y_1) = -x_1(y_1 - y_2) - y_1(x_2 - x_1) = x_1 y_2 + x_2 y_1 \end{aligned}$$

if we normalize such that  $\| (a, b) \| = 1$  then

$$d = ax + by + c$$

where d is a signed perpendicular distance from the line, positive to the left and negative to the right.

In fact the constraint imposed by the line can be discovered from the cross product of the two points:

$$L^T = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \vec{P}_1 \times \vec{P}_2 = \begin{pmatrix} 0 & -1 & y_1 \\ 1 & 0 & -x_1 \\ -y_1 & x_1 & 0 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \\ 1 \end{pmatrix} = \begin{pmatrix} y_1 - y_2 \\ x_2 - x_1 \\ x_1 y_2 - x_2 y_1 \end{pmatrix}$$

This is equivalent to a setting the determinant to zero with free variables in one of the rows.

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = x(y_1 - y_2) + y(x_2 - x_1) + x_1 y_2 - x_2 y_1 = 0$$

This is a simple example of a general principle that can be useful in higher dimensions.

## Two lines make a point

There is a perfect duality between points and lines. So naturally, two lines make a point.

Given non-parallel lines  $L$  and  $M$ .

$$L: a_1x + b_1y + c_1 = 0$$

and  $M: a_2x + b_2y + c_2 = 0.$

In classic notation  $L^T = (a_1, b_1, c_1)$  et  $M^T = (a_2, b_2, c_2)$

The unique point is  $x = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}$        $y = \frac{c_1a_2 - c_2a_1}{c_1a_2 - c_2a_1}$

The cross product is

$$\vec{P} = \vec{L} \times \vec{M} = \vec{P} = \vec{L} \times \vec{M} = \begin{pmatrix} 0 & -c_1 & b_1 \\ c_1 & 0 & -a_1 \\ -b_1 & a_1 & 0 \end{pmatrix} \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} = \begin{pmatrix} -c_1b_2 + b_1c_2 \\ c_1a_2 - a_1c_2 \\ -a_2b_1 + a_1b_2 \end{pmatrix} = \begin{pmatrix} b_1c_2 - c_1b_2 \\ c_1a_2 - a_1c_2 \\ a_1b_2 - a_2b_1 \end{pmatrix}$$

as before the cross product is equivalent to setting a determinant to zero. In this case:

$$\begin{vmatrix} a & b & c \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = a(b_1c_2 - c_1b_2) + b(c_1a_2 - a_1c_2) + c(a_1b_2 - b_1a_2) = 0$$

The unique point is  $x = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}$        $y = \frac{c_1a_2 - c_2a_1}{c_1a_2 - c_2a_1}$