Computer Vision

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M2R MoSIG option GVR

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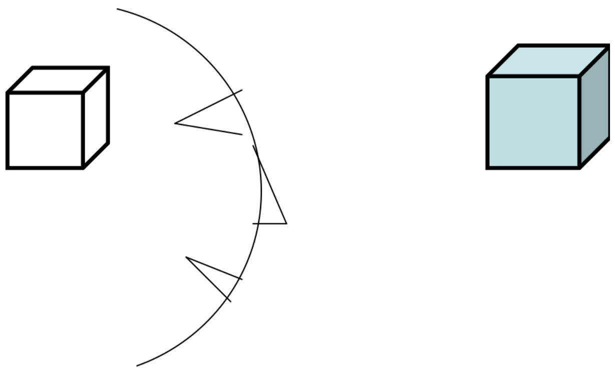
Lesson 4

Describing Images with Contrast Contours (Edges)

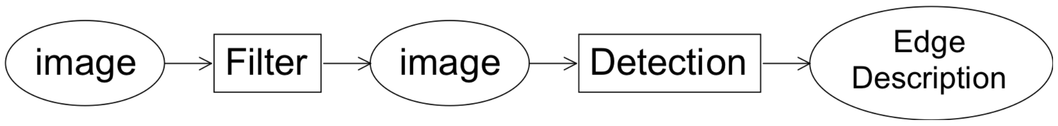
Lesson Outline:

1 Describing Images with Edges

For Lambertian reflection, changes in surface orientation result in changes in changes in image intensity. Abrupt changes in orientation cause abrupt changes in intensity. These changes are invariant to viewing position. For this reason, edges (contrast contours) became popular as a source of easily detected invariant image description during the 1960s and 1970s.



Edge detection is typically organized in three steps
1) contrast filtering
2) edge point detection,
3) edge point grouping and description.



A classic contrast detection operator is the Sobel edge detector.
Modern approaches use Gaussian Derivatives (made popular by J. Canny in 1986).

1.1 The Sobel Edge Detector

The "Sobel" edge detector was invented by Irwin Sobel in his 1964 Stanford Doctoral thesis. This detector was made famous by the classic text book on Pattern Recognition and Image Processing by R. Duda and P. Hart published in 1972.

The detector is composed of two "masks" which act as image filters.

$$m_r(i,j) = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & -2 \\ 1 & 0 & -1 \end{bmatrix} \qquad m_c(i,j) = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ -1 & -2 & -1 \end{bmatrix}$$

$m_c(i,j)$: detects contrast in column direction. *$m_r(i,j)$* : detects contrast in row direction

Convolution (or filtering) gives an edge vector: for n = r, c (row or column)

$$E_n(i,j) = m_n * p(i,j) = \sum_{k=-1}^{1}\sum_{l=-1}^{1} m_n(k,l)p(i-k,j-l) \qquad \vec{E}(i,j) = \begin{pmatrix} E_c(i,j) \\ E_r(i,j) \end{pmatrix} = \begin{pmatrix} m_c(i,j) \\ m_r(i,j) \end{pmatrix}$$

Attn: The "*" is NOT vector multiplication!! This is convolution.
Contrast is measured as the magnitude of edge energy:

$$E(i,j) = \|\vec{E}(i,j)\| = \sqrt{E_r(i,j)^2 + E_c(i,j)^2}$$

The direction of maximum contrast is the phase angle $\varphi(i,j) = Tan^{-1}\left(\frac{E_c(i,j)}{E_r(i,j)}\right)$

Sobel's edge filters can be seen as a composition of an image derivative and a binomial smoothing filter.

$$m_c(i,j) = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ -1 & -2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix} * \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = b_r(i,j) * d_c(i,j) = \sum_{k=-1}^{1}\sum_{l=-1}^{1} b_r(k,l)d_c(i-k,j-l)$$

$$m_r(i,j) = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & -2 \\ 1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} * \begin{bmatrix} 1 & 0 & -1 \end{bmatrix} = b_c(i,j) * d_r(i,j) = \sum_{k=-1}^{1}\sum_{l=-1}^{1} b_c(k,l)d_r(i-k,j-l)$$

The filter $b_c(i,j) = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ and $b_r(i,j) = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}$ are binomial smoothing filters.

The filters $d_c(i,j) = \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}$ and $d_r(i,j) = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ are a discrete form of image derivative.

1.2 Difference Operators: Derivatives for Sampled Signals

For continuous x, the derivative of a function s(x) can be defined as:

$$\frac{\partial s(x)}{\partial x} = \lim_{\Delta x \to 0} \left\{ \frac{s(x+\Delta x) - s(x-\Delta x)}{2\Delta x} \right\}$$

For a sampled signal, s(n), an the equivalent is $\frac{\Delta s(n)}{\Delta n}$:

$$\frac{\Delta s(n)}{\Delta n} = \frac{s(n+1) - s(n-1)}{2} = s(n) * \begin{bmatrix} -1/2 & 0 & 1/2 \end{bmatrix}$$

The ½ term is a scalar that can be neglected or normalized.

The result is the image difference (edge) operator used by Sobel.

$$\Delta n = 1 : \quad \frac{\Delta s(n)}{\Delta n} = s(n) * \begin{bmatrix} -1 & 0 & 1 \end{bmatrix}$$

1.3 Fourier Analysis of Derivative Operators

Note that a derivative is equivalent to a convolution!

We can define derivation in the Fourier domain as follows:

$$F\left\{ \frac{\partial s(x)}{\partial x} \right\} = -j\omega \cdot F\{s(x)\}$$

and thus

$$\frac{\partial s(x)}{\partial x} = F^{-1}\{-j\omega\} * s(x)$$

If we can determine $d(x) = F^{-1}\{-j\omega\}$ then we have our derivative operator.
If we "sample" d(x) to produce d(n) we have a sampled derivative operator.

Unfortunately, $F^{-1}\{-j\omega\}$ has an infinite duration in x, and thus d(n) is an infinite series. However, the first term of d(n) is [-1 0 1].

The Fourier Transform of a discrete signal is periodic in frequency with period π.

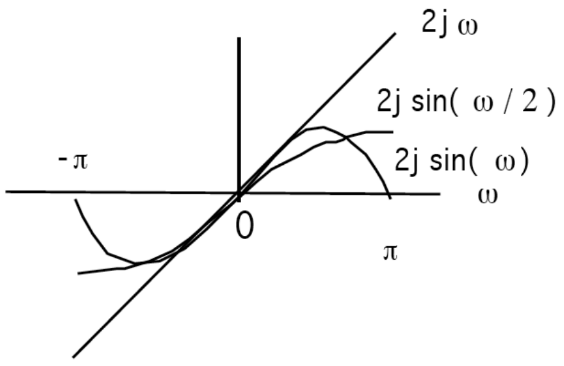
$$F\{s(n)\} = S(\omega) = \sum_{n=-\infty}^{\infty} s(n)e^{-j\omega n}$$

The first difference filter $d_1(n)$ = [1, 0, -1] has a Fourier transform:

$$D(\omega) = \sum_{n=-1}^{1} d(n)e^{-j\omega n}$$
$$D(\omega) = (1/2)e^{-j\omega(-1)} + 0e^{-j\omega 0} + (-1/2)e^{-j\omega(1)}$$
$$D(\omega) = (1/2)e^{j\omega} - (1/2)e^{-j\omega}$$
$$D(\omega) = -j\sin(\omega)$$

Calculation of a derivative is the same as convolution with the filter [1, 0, -1], which is the same as multiplication of the spectrums.

$$d(n) * s(n) \quad \Leftrightarrow D(\omega) \cdot S(\omega)$$



The derivative used by Sobel attenuates aliasing noise in high frequencies.

1.4 Smoothing: The Binomial Low pass filter.

Sobel uses a smoothing filter b(m) = [1, 2, 1] to overcome sampling noise.
This is a member of a family of filters generated by the binomial series.
The binomial series is the series of coefficients of the polynomial:

$$(x+y)^n = \sum_{m=0}^{n} b_{m,n} x^{n-m} y^m$$

The coefficients can be computed as $b_{m,n} = b_n(m) = [1, 1]^{*n}$

These are the coefficients of Pascal's Triangle.
The variable n, the number of "auto-convolutions" represents the "row" of the filter.
The value μ is the center of gravity of the filter. The value σ is the standard deviation

n	sum = 2^n	μ= n/2	σ^2 = n/4	$\sigma = \sqrt{n/2}$	Coefficients
0	1	0	0	0	1
1	2	0.5	0.25		1 1
2	4	1	0/5		1 2 1
3	8	1.5	0.75		1 3 3 1
4	16	2	1	1	1 4 6 4 1
5	32	2.5	1.25		1 5 10 10 5 1
6	64	3	1.5		1 6 15 20 15 1
7	128	3.5	1.75		1 7 21 35 35 21 7 1
8	256	4	2	$\sqrt{2}$	1 8 29 56 70 56 29 8 1

These coefficients provide a family of low pass filters with remarkable properties. They have integer coefficients, and thus are cheap to convolve on an embedded system (no floating point required). They are also the best finite length approximation for a Gaussian filter.

$$b_n(m) = [\,1,\ \ 1]^{*n} = \text{n convolutions of } [1,\ 1]$$

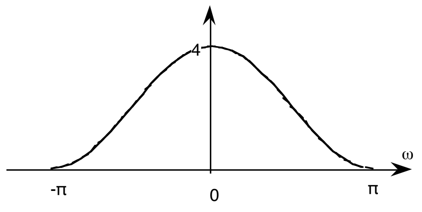
Gain (sum): $s_n = \sum_{m=1}^{n} b_n(m) = 2^n$

Center of gravity is $\mu_n = \frac{1}{s_n}\sum_{m=1}^{n} b_n(m)\cdot m = \frac{n}{2}$

The variance is: $\sigma_n^2 = \frac{1}{s_n}\sum_{m=1}^{n} b_n(m)\cdot(m-\mu)^2 = \frac{n}{4}$

The Fourier transform for $b_2(m) = [1, 2, 1]$ is

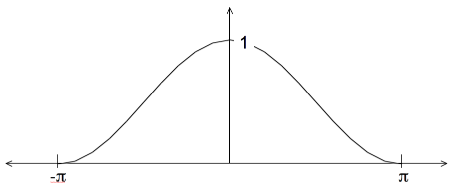
$$B_2(\omega) = \sum_{m=-1}^{1} b_2(m) e^{-j\omega m}$$
$$B_2(\omega) = 1e^{-j\omega(-1)} + 2e^{-j\omega 0} + 1e^{-j\omega(1)}$$
$$B_2(\omega) = 2 + e^{j\omega} + e^{-j\omega}$$
$$B_2(\omega) = 2 + 2\cos(\omega)$$



If we normalize the gain: $b_2(m) = (1/4)[1, 2, 1]$

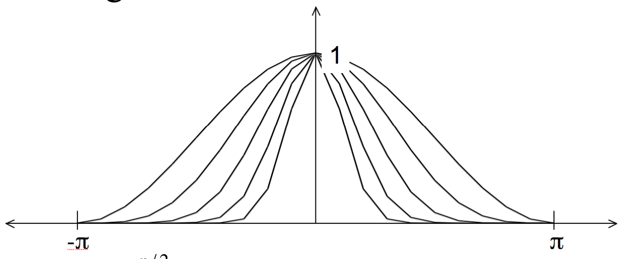
$$B_2(\omega) = \frac{1}{2} + \frac{1}{2}\cos(\omega)$$

Which is a cosine on a platform



$$B_2(\omega) = \frac{1}{2} + \frac{1}{2}\cos(\omega)$$

Repeated convolution makes generates



$$B_n(\omega) = \left(\frac{1}{2} + \frac{1}{2}\cos(\omega)\right)^{n/2}$$

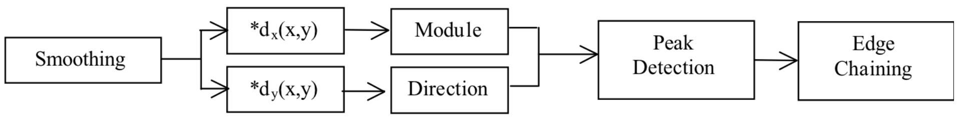
The binomial coefficients provide a series of low pass filters with no ripples.
In 2D, the filters provide separable filters that are nearly circularly symmetric

$$\text{2-D} \quad b2(i, j) = \boxed{\begin{matrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{matrix}} = \boxed{\begin{matrix} 1 \\ 2 \\ 1 \end{matrix}} * \boxed{\begin{matrix} 1 & 2 & 1 \end{matrix}}$$

2 Describing Contrast with Edges

Contrast is a change in image intensity.
Classically, contrast was described by detecting edges.

2.1 Edge Detection using integer coefficient filters



The Gradient of the image is a vector of derivatives: $\vec{\nabla} P(i,j) = \begin{pmatrix} p_i(i,j) \\ p_j(i,j) \end{pmatrix}$

where $p_i(i,j) = \frac{\Delta p(i,j)}{\Delta i} = p(i,j) * \begin{bmatrix} -1 & 0 & 1 \end{bmatrix}$ $\qquad p_j(i,j) = \frac{\Delta p(i,j)}{\Delta j} = p(i,j) * \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

Gradient magnitude indicates edge strength $E(i,j) = \|\vec{\nabla} P(i,j)\|$

Gradient direction is the direction of maximum contrast

As an angle: $\vartheta(i,j) = Tan^{-1}\left(\frac{p_j(i,j)}{p_i(i,j)}\right)$

As a vector: $\vec{d}(i,j) = \begin{pmatrix} Sin(\vartheta(i,j)) \\ Cos(\vartheta(i,j)) \end{pmatrix} = \frac{1}{\|\nabla P(i,j)\|} \begin{pmatrix} p_i(i,j) \\ p_j(i,j) \end{pmatrix} = \begin{pmatrix} \frac{p_i(i,j)}{\|\nabla P(i,j)\|} \\ \frac{p_j(i,j)}{\|\nabla P(i,j)\|} \end{pmatrix}$

Note that we can estimate the direction of maximum gradient, $\vartheta(i,j)$, by normalizing the derivatives along rows and columns.

2.2 Non-maximum suppression.

Contrast points C(i, j) are local maxima in $\vec{\nabla} P(i,j)$ in the direction of maximum gradient. $\vartheta(i,j)$

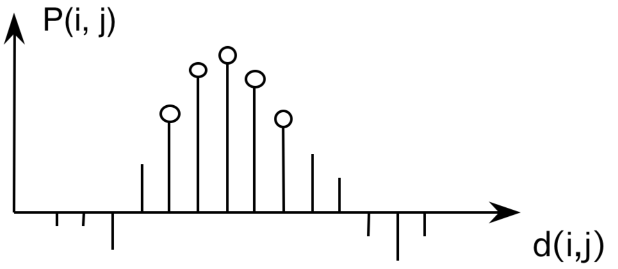
$$\vec{\nabla} P(i,j) = \begin{pmatrix} p_i(i,j) \\ p_j(i,j) \end{pmatrix} \qquad \vartheta(i,j) = Tan^{-1}\left(\frac{\Delta_j p(i,j)}{\Delta_i p(i,j)}\right) = Tan^{-1}\left(\frac{E_j(i,j)}{E_i(i,j)}\right)$$

For each edge pixel:

1) Determine the direction of maximum gradient:

$$\Delta i = \cos(\vartheta(i,j)) = \frac{P_i(i,j)}{\|\vec{\nabla} P(i,j)\|} \qquad \Delta j = \sin(\vartheta(i,j)) = \frac{P_j(i,j)}{\|\nabla P(i,j)\|}$$

2) Compare the gradient to its neighbors in this direction.



$$c(i,j) = \begin{cases} 1 & \text{IF } E(i-\Delta i, j-\Delta j) \le E(i,j) \ge E(i+\Delta i, j+\Delta j) \\ 0 & \text{Otherwise} \end{cases}$$

This requires rounding the values Δi and Δj to obtain integer values.

Alternatively, we can consider $E(i,j) = \|\vec{\nabla} P(i,j)\|$ as "evidence" for contrast at all points

3 The Hough Transform

The Hough transform is an "optimal" statistical detector for estimating parametric functions from discrete samples. This method was invented for interpreting bubble chamber images in particle physics. It is based on "voting" for possible parameters.

This transform was invented by
P.V.C. Hough, Machine Analysis of Bubble Chamber Pictures, Proc. Int. Conf. High Energy Accelerators and Instrumentation, 1959

It was patented in a crude form by IBM in 1962 using $y = mx+c$.

It was made popular by Duda and Hart :
Duda, R. O. and P. E. Hart, "Use of the Hough Transformation to Detect Lines and Curves in Pictures," Comm. ACM, Vol. 15, pp. 11–15 (January, 1972)

3.1 The Algorithm

Consider the line equation

$$x \cos(\theta) + y \sin(\theta) + c = 0$$

In the image, for each x,y (free parameters) we need to determine (c, θ)

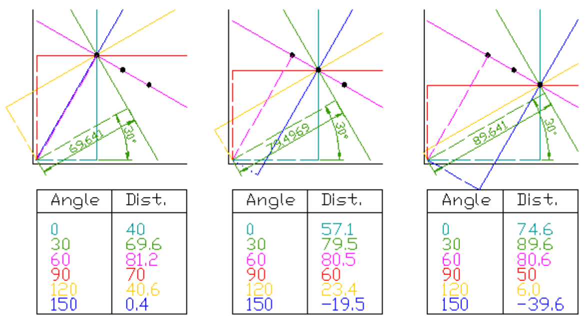
In the Hough transform, we will create a dual space in which (c, θ) are free parameters.
We will estimate lines as peaks in this dual space. To find peaks we build an accumulator array : $h(c, \theta)$.

Let the c be an integer $c \in [0, D]$ where D is the "diagonal distance of the image.
Let θ be an integer $\theta \in [0, 179]$

Algorithm:
```
allocate a table  $h(c, \theta)$ initially set to 0.
For each x, y of the image
    for  $\theta$ from 0 to 179
        $c = -x \cos(\theta) - y\sin(\theta)$
        $h(c, \theta) = h(c, \theta) + E(x, y)$
    End
End
```

The resulting table accumulates contrast.

Peaks in h(c, θ) correspond to line segments in the image.

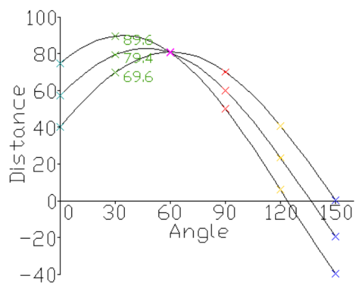


Angle	Dist.
0	40
30	69.6
60	81.2
90	70
120	40.6
150	0.4

Angle	Dist.
0	57.1
30	79.5
60	80.5
90	60
120	23.4
150	−19.5

Angle	Dist.
0	74.6
30	89.6
60	80.6
90	50
120	6.0
150	−39.6

=>



Because we know θ(x, y), we can limit the evaluation to θ(x, y)+/- Δθ

3.2 Generalization of the Hough Transform

We can represent a circle with the equation:

$$(x - a)^2 + (y - b)^2 = r^2$$

We can use this to create a Hough space h(a, b, r) for limited ranges of r.

The ranges of a and b are the possible positions of circles.

Algorithm

Algorithm:

```
allocate a table  h((a, b, r) initially set to 0.
For each x, y of the image
    for r from $r_{min}$ to $r_{max}$
        for a from 0 to $a_{max}$
            $b = -y - \mathrm{sqrt}( r^2 - (x - a)^2)$
            $h(a,b,r) = h(a,b,r) + E(x,y)$.
        End
    End
End
```

4 Image Description Using Gaussian Derivatives

4.1 The Gaussian function

The Gaussian function is: $G(x,\sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}$

Fourier Transform: $F\{e^{-\frac{x^2}{2\sigma^2}}\} = G(\omega,\sigma) = \sqrt{2\pi}\ \sigma\, e^{-\frac{1}{2}\sigma^2\omega^2}$

Scale property: $G(x,\sqrt{2}\sigma) = G(x,\sigma) * G(x,\sigma)$

Derivatives: $\frac{\partial G(x,\sigma)}{\partial x} = G_x(x,\sigma) = -\frac{x}{\sigma^2} G(x,\sigma)$

$$\frac{\partial^2 G(x,\sigma)}{\partial x^2} = G_{xx}(x,\sigma) = \frac{x^2-\sigma^2}{\sigma^4} G(x,\sigma)$$

$$\frac{\partial^3 G(x,\sigma)}{\partial x^3} = G_{xxx}(x,\sigma) = \frac{x^3-x\sigma^2}{\sigma^6} G(x,\sigma)$$

4.2 2D Gaussian functions

2D Gaussian Kernel: $G(x,y,\sigma) = \frac{1}{2\pi\sigma^2} e^{-\frac{(x^2+y^2)}{2\sigma^2}}$

Fourier Transform: $F\{e^{-\frac{x^2+y^2}{2\sigma^2}}\} = \frac{\pi}{2\sigma^2} e^{-\frac{1}{2}\sigma^2(u^2+v^2)}$

Separability: $G(x,y,\sigma) = \frac{1}{2\pi\sigma^2} e^{-\frac{(x^2+y^2)}{2\sigma^2}} = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}} * \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{y^2}{2\sigma^2}}$

Derivatives: $\frac{\partial G(x,y,\sigma)}{\partial x} = G_x(x,y,\sigma) = -\frac{x}{\sigma^2} G(x,y,\sigma)$

$$\frac{\partial G(x,y,\sigma)}{\partial y} = G_y(x,y,\sigma) = -\frac{y}{\sigma^2} G(x,y,\sigma)$$

$$\frac{\partial^2 G(x,y,\sigma)}{\partial x^2} = G_{xx}(x,y,\sigma) = \frac{x^2-\sigma^2}{\sigma^4} G(x,y,\sigma)$$

$$\frac{\partial^2 G(x,y,\sigma)}{\partial x \partial y} = G_{xy}(x,y,\sigma) = \frac{xy}{\sigma^4} G(x,y,\sigma)$$

$$\frac{\partial^3 G(x,y,\sigma)}{\partial x^3} = G_{xxx}(x,y,\sigma) = \frac{x^3-x\sigma^2}{\sigma^6} G(x,y,\sigma)$$

4.3 Using the Gaussian to compute image derivatives

For an image *p(i,j)*, the derivatives can be approximated by convolution with Derivatives of Gaussians.

$$\frac{\partial p(i,j)}{\partial x} * G(x,y) = \frac{\partial}{\partial x} * p(i,j) * G(x,y) = \frac{\partial}{\partial x} * G(x,y) * p(i,j) = \frac{\partial G(x,y)}{\partial x} * p(i,j)$$

Thus we can approximate an image derivative as $P_x(i,j) \approx G_x * P(i,j)$
However to compute G_x, it is NECESSARY to specify σ.
Small σ is not necessarily best.

$$p_x(i,j,\sigma) \approx G_x(x,y,\sigma) * p(i,j)$$

or more simply $p_x(i,j,\sigma) \approx G_x(\sigma) * p(i,j)$

Similarly: $p_y(i,j,\sigma) \approx G_y(\sigma) * p(i,j)$

$$p_{xx}(i,j,\sigma) \approx G_{xx}(\sigma) * p(i,j)$$
$$p_{xy}(i,j,\sigma) \approx G_{xy}(\sigma) * p(i,j)$$
$$p_{yy}(i,j,\sigma) \approx G_{yy}(\sigma) * p(i,j)$$

The Gradient of the image $\vec{\nabla} p(i,j)$ is calculated by $\vec{\nabla} G(\sigma) * p(i,j)$

where $\vec{\nabla} G(\sigma) = \begin{pmatrix} G_x(\sigma) \\ G_y(\sigma) \end{pmatrix}$ This gives:

Gradient: $\vec{\nabla} p(i,j,\sigma) = \begin{pmatrix} p_x(i,j,\sigma) \\ p_y(i,j,\sigma) \end{pmatrix} \approx \vec{\nabla} G(\sigma) * p(i,j) = \begin{pmatrix} G_x(\sigma) \\ G_y(\sigma) \end{pmatrix} * p(i,j)$

Laplacien:

$$\nabla^2 p(i,j,\sigma) = \nabla^2 G(\sigma) * p(i,j) = p_{xx}(i,j,\sigma) + p_{yy}(i,j,\sigma) \approx G_{xx}(\sigma) * p(i,j) + G_{yy}(\sigma) * p(i,j)$$

4.4 The Laplacian of the Gaussian and the DoG

The Laplacian of Gaussian is a scalar value:

$$\nabla^2 G(x,y,\sigma) = G_{xx}(x,y,\sigma) + G_{yy}(x,y,\sigma) = \frac{\partial G(x,y,\sigma)}{\partial \sigma}$$

Because it is the derivative with respect to s, it can be approximated by a difference of Gaussians (DoG) :

$$\nabla^2 G(x,y,\sigma) \approx G(x,y,\sigma_1) - G(x,y,\sigma_2)$$

This is called a Difference of Gaussian and typically requires $\sigma_1 \geq 1.4\ \sigma_2$

It is common to use: $\nabla^2 G(x,y,\sigma) \approx G(x,y,\sqrt{2}\sigma) - G(x,y,\sigma)$

Because of the scale property: $G(x,y,\sqrt{2}\sigma) \approx G(x,y,\sigma) * G(x,y,\sigma)$

We can easily compute a DoG as

$$\nabla^2 G(x,y,\sigma) \approx G(x,y,\sigma) * G(x,y,\sigma) - G(x,y,\sigma)$$